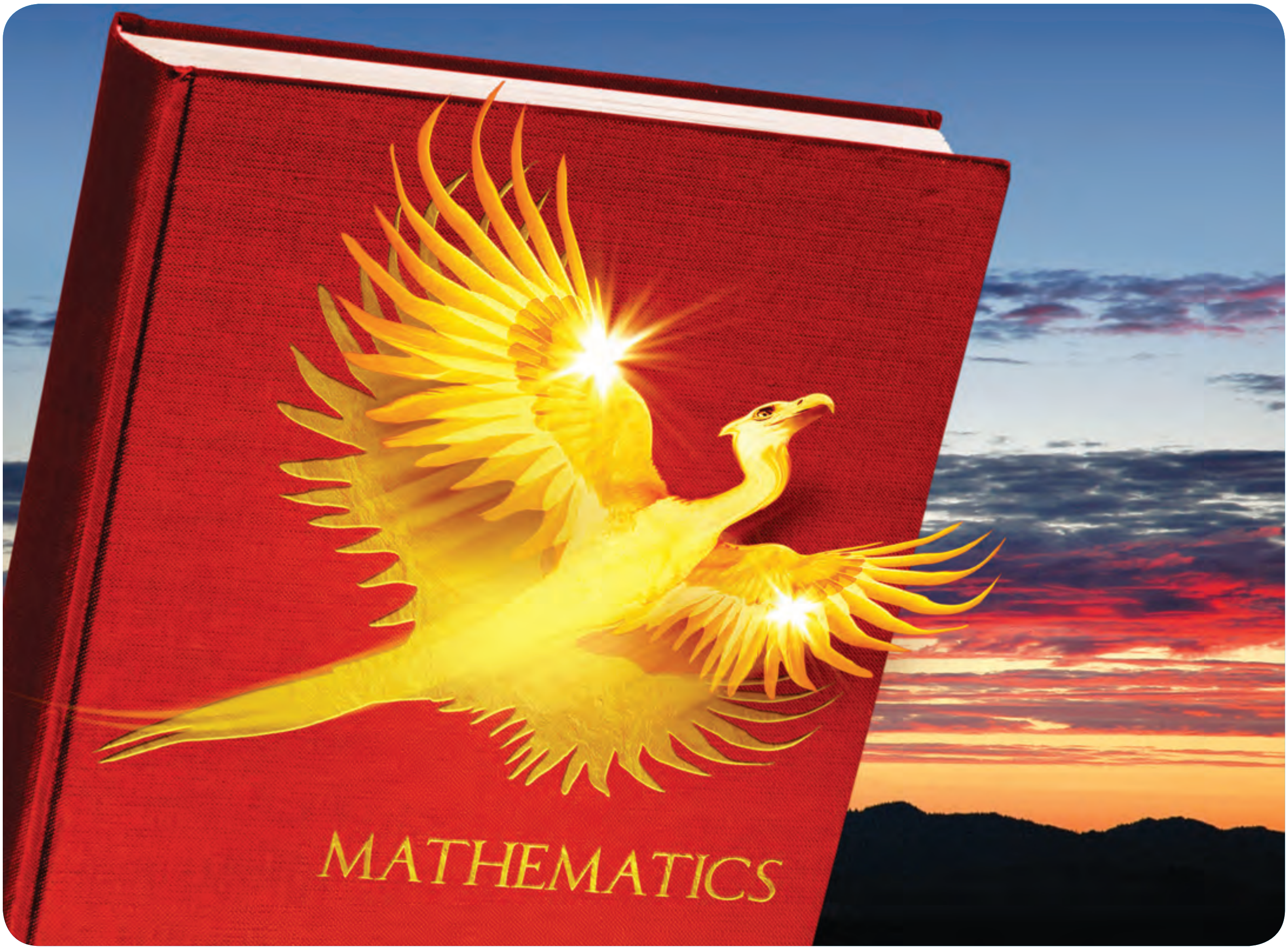
Phoenix Rising

Bringing the Common Core State Mathematics Standards to Life



By Hung-Hsi Wu

Many sets of state and national mathematics standards have come and gone in the past two decades. The Common Core State Mathematics Standards (CCSMS), which were released in June of 2010,* have been adopted by almost all states and will be phased in across the nation in 2014. Will this be another forgettable standards document like the overwhelming majority of the others?

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Perhaps. But unlike the others, it will be a travesty if this one is forgotten. The main difference between these standards and most of the others is that the CCSMS are mathematically very sound overall. They could serve—at long last—as the foundation for creating proper school mathematics textbooks and dramatically better teacher preparation.

Before the CCSMS came along, America long resisted the idea of commonality of standards and curriculum—but it did not resist such commonality in actual classrooms. Despite some politicians' rhetoric extolling the virtues of local control, there has been a de facto national mathematics curriculum for decades: the curriculum defined by the school mathematics textbooks. There are several widely used textbooks, but mathematically they are very much alike. Let's call this de facto mathematics curriculum Textbook School Mathematics (TSM).[1] In TSM, precise definitions usually are not given and logical rea-

*To learn more about the Common Core State Mathematics Standards, see www.corestandards.org.

ILLUSTRATIONS BY WILLIAM DUKE

soning is hardly ever provided (except in high school geometry texts) because the publishers mistakenly believe that intuitive arguments and analogies suffice. Thus, fractions are simultaneously (and incomprehensibly) parts of a whole, a division, and a ratio; decimals are taught independently from fractions by appealing to the analogy with whole numbers; negative numbers are taught by using patterns and metaphors; the central idea of beginning algebra is the introduction of the concept of a *variable* (which implies, wrongly, that something is going to vary), when it ought to be becoming fluent in using symbols so as to do generalized arithmetic; solving equations is explained by the use of a balance to weigh variables on the weighing platforms; etc.

Worse, with TSM in the background, the prevailing dogma in mathematics education is that the main purpose of a set of standards is either to pick and choose from a collection of tried-and-true topics (from TSM, of course) and organize the selected items judiciously, or to vary the pedagogical approaches to these topics. For example, when California's Number Sense Standards ask that, in grade 5, "Students perform calculations and solve problems involving addition, subtraction, and simple multiplication and division of fractions and decimals," it is understood that all of the classrooms will do these arithmetic operations on fractions in accordance with TSM. From this perspective, the main point of this standard is that these calculations with fractions are taught *in the fifth grade*. Indeed, the very purpose of mathematics standards (prior to the CCSMS) seems to be to establish in which grade topics are to be taught. Often, standards are then judged by how early topics are introduced; thus, getting addition and subtraction of fractions done in the fifth grade is taken as a good sign. By the same ridiculous token, if a set of standards asks that the multiplication table be memorized at the beginning of the third grade or that Algebra I be taught in the eighth grade, then it is considered to be *rigorous*.

The CCSMS challenge this dogma. Importantly, the CCSMS do not engage in the senseless game of acceleration—to teach every topic as early as possible—even though refusing to do so has been a source of consternation in some quarters. For example, the CCSMS do not complete all the topics of Algebra I in grade 8 because much of the time in that grade is devoted to the geometry that is needed for understanding the algebra of linear equations.[2] But the real contribution of the CCSMS lies in their insistence on righting the many wrongs in TSM. As opposed to the standards of years past, *the CCSMS are aware of the chasm between what TSM is and what school mathematics ought to be*. They are unique in their realization that the flaws in the logical development of most topics in TSM—not how early or how late each topic is placed in the standards—are the real impediment to any improvement in mathematics education. *Garbage in, garbage out*, as the saying goes. If we want students to learn mathematics, we have to teach it to them. Neither the previous mathematics standards nor the TSM on which they rely did that, but the CCSMS do.

Beyond the frequent absence of reasoning, the disconnectedness in the presentation of mathematical topics in TSM turns a coherent subject into nothing more than a bag of tricks. Students are made to feel that what is learned one year can be forgotten in the next. By contrast, the CCSMS succeed in most instances in maintaining continuity from grade to grade. The most striking example may well be the seamless transition from eighth-grade geometry to high school geometry. In fact, the CCSMS succeed in integrating geometry into the overall fabric of school mathematics. The mathematics in the CCSMS finally begins to look like *mathematics*.

Unfortunately, textbook developers have yet to accept that the CCSMS are radically different from their predecessors. Most (and possibly all) textbook developers are only slightly revising their texts before declaring them aligned with the CCSMS. Do not be fooled. TSM is much too vague and has far too many errors to be aligned with the CCSMS. For example, when the National Mathematics Advisory Panel reviewed two widely used algebra textbooks to determine their "error density" (which was defined as the number of errors divided by the number of pages in the book), it found that one had an error density of 50 percent and the other was only slightly better at 41 percent.[3] We must start from scratch. Since teacher education in mathematics has long been based on TSM, both pre-service and in-service training must also be created anew.

Let us give two examples of the kind of change the CCSMS (if properly implemented) will bring to the mathematics classroom.

Example 1: Adding Fractions

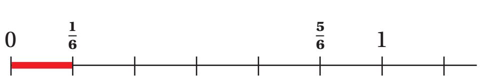
How should students add $\frac{1}{8} + \frac{5}{6}$? The way it is done in TSM is to not say anything about what it means to add fractions, but instead to prescribe the procedure of finding the least common multiple of the denominators 8 and 6, which is 24, and note that $24 = 3 \times 8$ and $24 = 4 \times 6$. Students are then instructed to add as follows:

$$\frac{1}{8} + \frac{5}{6} = \frac{(3 \times 1)}{(3 \times 8)} + \frac{(4 \times 5)}{(4 \times 6)} = \frac{23}{24}.$$

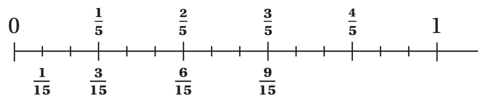
By all accounts, this procedure makes no sense to fifth-graders, but many seem to memorize it and it remains firmly entrenched in TSM. Adding is supposed to "combine things." The concept of "combining" is so basic that it is always taught at the beginning of arithmetic. Yet, can one detect any "combining" in the TSM approach to $\frac{1}{8} + \frac{5}{6}$? Children who have made the effort to master the addition of whole numbers naturally expect that the *addition* of fractions will be more of the same, i.e., "combining things." But when "adding fractions" is presented as having nothing to do with "adding whole numbers," the fear that they cannot articulate is undoubtedly that mathematics is impossible to understand. Indeed, there are reports that much math phobia begins with adding fractions.

Unfortunately, textbook developers have yet to accept that the Common Core State Mathematics Standards are radically different from their predecessors.

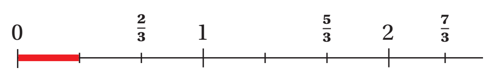
In the CCSMS, adding fractions is spread through three grades, progressing from the simple to the complex, giving students time for complete mastery.* Briefly, in grade 3, students learn to think of a fraction as a point on the number line that is "so many copies" of its corresponding *unit fraction*. For example, $\frac{5}{6}$ is 5 copies of the unit fraction $\frac{1}{6}$ (and $\frac{1}{6}$ is 1 copy). When we represent a fraction as a point on the number line, we place a unit fraction such as $\frac{1}{6}$ on the division point to the right of 0 when the *unit segment* from 0 to 1 is divided into 6 equal segments. It is natural to identify such a point with the segment between the point itself and 0. Thus, as shown below, $\frac{1}{6}$ is identified with the red segment between 0 and $\frac{1}{6}$, $\frac{5}{6}$ is identified with the segment between 0 and $\frac{5}{6}$, etc. Then, the statement that "$\frac{5}{6}$ is 5 copies of $\frac{1}{6}$" acquires an obvious visual meaning: the segment from 0 to $\frac{5}{6}$ is 5 copies of the segment from 0 to $\frac{1}{6}$.



In third grade, students also learn about simple cases of equivalent fractions: $\frac{2}{5}$ is the same point as—i.e., *is equal to*—$(3 \times 2)/(3 \times 5)$, or $\frac{6}{15}$. This is because $\frac{2}{5}$ is the second division point to the right of 0 when the unit segment from 0 to 1 is divided into 5 equal segments. When each of these 5 segments is divided into 3 equal segments, it creates a division of the unit segment into $3 \times 5 = 15$ equal segments. It is then obvious that the point $\frac{2}{5}$ is exactly the same point as $\frac{6}{15}$, which is $(3 \times 2)/(3 \times 5)$, as shown below.

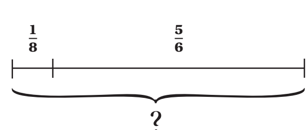


In grade 4, the CCSMS call for students to learn about adding two fractions as *joining two parts* of the same whole. Think of the two fractions as segments, put them together end-to-end on the same number line, and the sum is by definition the length of the joined segment. For fractions with the same denominator, adding these fractions yields a fraction whose numerator is the sum of the respective numerators, as we can see clearly from an example. Let's show that $\frac{2}{3} + \frac{5}{3} = (2 + 5)/3$. On the number line below, $\frac{1}{3}$ is the red segment.



Thus, $\frac{2}{3}$ is 2 copies of the red segment and $\frac{5}{3}$ is 5 copies of the red segment, so "combining" $\frac{2}{3}$ and $\frac{5}{3}$ yields (2+5) copies of the red segment, which is $\frac{7}{3}$. Therefore, adding fractions *is* "combining things" in this case.†

In grade 4, students also go beyond the simple cases to learn about equivalent fractions in general. Then in the fifth grade of the CCSMS, students handle the sum of any two fractions. Of course, it is still obtained by *joining parts*: putting two segments together so that the sum is the total length. This they are ready for because, by use of equivalent fractions, any two fractions may be regarded as two fractions with the same denominator. For example, $\frac{1}{8}$ and $\frac{5}{6}$ are equal to $(6 \times 1)/(6 \times 8)$ and $(8 \times 5)/(8 \times 6)$, which now have the same denominator, 48. So these fifth-graders can easily address our original question—How should students add $\frac{1}{8} + \frac{5}{6}$?—in a mathematically sound manner. With their strong foundation from the third and fourth grades of the CCSMS, they know that this addition problem is the same as asking how long the following combined segment is.



So, $$\frac{1}{8} + \frac{5}{6} = \frac{(6 \times 1)}{48} + \frac{(8 \times 5)}{48} = \frac{[(6 \times 1) + (8 \times 5)]}{48} = \frac{46}{48}.$$

*For an extended discussion of how to teach fractions in grades 3–7 in accordance with the CCSMS, please see my guide "Teaching Fractions According to the Common Core Standards," available at http://math.berkeley.edu/~wu/CCSS-Fractions.pdf.

†For an extended discussion of how to approach these two examples from the point of view of the number line, one may consult parts 2 and 3 of my new textbook for teachers, *Understanding Numbers in Elementary School Mathematics*, published by the American Mathematical Society. (See the box on pages 12–13.)

This is the same answer as before because, by equivalent fractions, $46/48 = 23/24$. Therefore, students get to see that adding fractions *is* "combining things." Incidentally, there has been no mention of the least common multiple of 8 and 6, and this is as it should be. (My pointing out that $46/48 = 23/24$ should not be interpreted as affirming the common practice of insisting that every fraction be reduced to the simplest form. There is no mathematical justification for this practice; I did it merely to show that we got the same answer either way.)

I hope this example begins to clarify the vast differences between TSM and the CCSMS. Adding fractions is a foundational topic: TSM gives students (and teachers) a gimmick; the CCSMS require that students actually learn mathematics.

Example 2: Multiplying Negative Numbers

Why is $(-2)(-3) = 2 \times 3$? This is quite possibly the most frequently asked question in K–12 mathematics: why is negative times negative positive? The answer, according to TSM, can be given in terms of patterns. For the specific case of $(-2)(-3)$, we observe that the values of $4(-3)$, $3(-3)$, $2(-3)$, $1(-3)$, and $0(-3)$ are as follows:

$$4(-3) = (-3) + (-3) + (-3) + (-3) = -12$$
$$3(-3) = (-3) + (-3) + (-3) = -9$$
$$2(-3) = (-3) + (-3) = -6$$
$$1(-3) = -3$$
$$0(-3) = 0.$$

There is an unmistakable pattern: the answer on each line is obtained by adding 3 to the answer from the line above. Thus, starting with the last line, $0 = 3 + (-3)$, $-3 = 3 + (-6)$, $-6 = 3 + (-9)$, $-9 = 3 + (-12)$, and of course the pattern persists if we also take into account $5(-3)$, $6(-3)$, etc. But if we now continue the sequence of multiplications of $4(-3)$, $3(-3)$, $2(-3)$, $1(-3)$, and $0(-3)$, then the next couple of items in line will be

$$(-1)(-3) = ?$$
$$(-2)(-3) = ?$$

Encouraged by the pattern we just observed, we are confident that the number $(-1)(-3)$ should be one that is obtained from the number $0(-3)$ (which is 0) by adding 3: $(-1)(-3) = 3 + 0 = 3$. Similarly, $(-2)(-3)$ should be one obtained from $(-1)(-3)$ by adding 3: $(-2)(-3) = 3 + 3 = 2 \times 3$.

Is this a good explanation? No. There are two problems. First, if instead of dealing with the product of integers, we consider a product such as $(-5/11)(-4/3)$, then a little thought would reveal that this reasoning by patterns breaks down completely. Second, we must convince ourselves that the pattern *should* persist all the way to $(-1)(-3)$, $(-2)(-3)$, $(-3)(-3)$, etc. In greater detail, this pattern asks students to believe that

$$(-1)(-3) = 3 + 0(-3),$$
$$(-2)(-3) = 3 + (-1)(-3),$$
$$(-3)(-3) = 3 + (-2)(-3), \text{ etc.}$$

Of these, the critical one is the first: $(-1)(-3) = 3$. If we know that, then, with or without a pattern, we will have the remaining equalities for the following reason. The *distributive law*, which is a statement about how multiplication behaves with respect to addition, says if x, y, and z are any three numbers, we always have $[y + z]\,x = yx + zx$. Thus, for example, $[2 + (-1/3)](-4) = 2(-4) + (-1/3)(-4)$. The fact that all numbers positive or negative obey the distributive law is a fundamental assumption in mathematics. Now if $y = z = (-1)$ and $x = (-3)$, then we have $[(-1) + (-1)](-3) = (-1)(-3) + (-1)(-3)$. Making use of this fact and *assuming* $(-1)(-3) = 3$, we now get:

$$(-2)(-3) = [(-1) + (-1)](-3) = (-1)(-3) + (-1)(-3) = 3 + 3 = 2 \times 3.$$

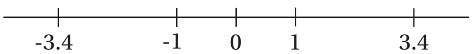
For exactly the same reason, we would get $(-3)(-3) = 3 \times 3$, $(-4)(-3) = 4 \times 3$, etc., provided we assume $(-1)(-3) = 3$. But how do we know $(-1)(-3) = 3$? In TSM, there is no answer. This is the nature of TSM: it often *half*-satisfies students' appetite for knowledge—but given the precise nature of mathematics, this is almost the same as no knowledge at all.

Let us now look at what the CCSMS say on this matter. In the broader context of understanding negative numbers, it is important that students have a clear conception of what a negative number is. It should be a specific object rather than some ineffable philosophical idea. For this, the CCSMS go back to the number line just as in the case of fractions.* One standard in the CCSMS for grade 6 has this to say:

*This is a small example of the longitudinal coherence of mathematics: the fact that fractions and rational numbers are united by the number line.

> Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g., $-(-3) = 3$, and that 0 is its own opposite.

Negative numbers are points on the number line to the left of 0. More precisely, for each fraction that is a point to the right of 0, its negative is the point to the left of 0 that is equidistant from 0. We can think of a fraction such as 3.4 (which is $^{34}/_{10}$, by definition) and its negative -3.4 as *mirror images* of each other with respect to 0, as shown below.



Textbook School Math often *half*-satisfies students' appetite for knowledge—but given the precise nature of mathematics, this is almost the same as no knowledge at all.

Jumping ahead to multiplying negative numbers, the CCSMS for grade 7 say the following:

> Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such as $(-1)(-1) = 1$ and the rules for multiplying signed numbers. Interpret products of rational numbers by describing real-world contexts.

This standard needs amplification, which I will provide in the process of giving a correct explanation of $(-2)(-3) = 2 \times 3$. This explanation will be valid also for the general case of $(-m)(-n) = mn$ for any integers m and n. When m and n are fractions (which is what this standard calls for), a slightly more sophisticated explanation will be necessary (and thus should be provided by any decent textbook), but we will settle for the simpler case here.

The key step in the correct explanation lies in the proof of $(-1)(-1) = 1$ (as asserted in the grade 7 standard). Pictorially, what this equality says is that multiplying (-1) by (-1) flips (-1) to its mirror image 1 on the right side of 0. A more expansive treatment of this topic in accordance with the CCSMS would show that, more generally, multiplying any number by (-1) flips it to its mirror image on the other side of 0.

Now, how to find out if $(-1)(-1)$ is the number 1 or not? For students in grades 6 or 7, the most desirable way to do so is by performing a direct computation that starts with $(-1)(-1)$ and ends with 1. However, since there is no known way of doing this, we'll take an indirect approach by anticipating the right answer (which is 1, of course) and asking: is $(-1)(-1) + (-1)$ equal to 0? If so, then we will see that $(-1)(-1)$ is equal to 1 and we are done. The key difference between $(-1)(-1)$ and the longer expression $(-1)(-1) + (-1)$ is that we can actually do a computation on the latter! We appeal to the distributive law in the second equal sign below.

$$(-1)(-1) + (-1) = (-1)(-1) + 1\,(-1) = [(-1) + 1]\,(-1) = 0\,(-1) = 0$$

Notice that it is only when we get to $[(-1)+1]\,(-1)$ that we can begin to "compute" in the usual sense of arithmetic: $(-1) + 1$ is equal to 0, and $0(-1)$ is also 0. In any case, we have finally demonstrated—using familiar arithmetic—that $(-1)(-1) = 1$.

Now we can prove $(-2)(-3) = 2 \times 3$. We first show $(-1)(-3) = 3$. We have $(-1)(-3) = (-1)[(-1) + (-1) + (-1)]$ which, by the distributive law again, is equal to $(-1)(-1) + (-1)(-1) + (-1)(-1) = 1 + 1 + 1 = 3$. Thus $(-1)(-3) = 3$. Having taken care of our earlier concern as to why $(-1)(-3)$ is equal to 3, we can now easily complete our reasoning about $(-2)(-3) = 2 \times 3$, namely: $(-2)(-3) = [(-1) + (-1)](-3) = (-1)(-3) + (-1)(-3)$, by the distributive law (yet again!). And, by what we just proved, the latter is $3 + 3 = 2 \times 3$. So $(-2)(-3) = 2 \times 3$ after all.

If we reflect on the reasoning above, we see clearly that the critical step was the application of the distributive law; without that it would have been impossible to conclude that $(-1)(-1) + (-1) = 0$, that $(-1)(-3) = 3$, or that $(-2)(-3) = 2 \times 3$. This is exactly the main emphasis in the preceding standard from the CCSMS. The proof of $(-m)(-n) = mn$, for whole numbers m and n, is entirely similar. Thus, a teacher guided by the CCSMS, unlike a teacher guided by TSM, would provide a correct and complete mathematical explanation of why a negative times a negative equals a positive. There is no need to look for patterns that do not hold true and no excuse for providing a half-satisfactory explanation.

It takes no real knowledge of mathematics to see from these two examples that the leap from TSM to the mathematical demands of the CCSMS is a gigantic one. With more space, I could provide many more examples: most of the time, the distance between TSM and the CCSMS is vast. We cannot expect the nation's teachers to implement the CCSMS on their own. So far, textbook developers are not rising to the challenge of the CCSMS. Our only hope, therefore, lies in providing professional development to help our teachers acquire the mathematical knowledge necessary to see the flaws in TSM.

"Start Selling What They Need"

For in-service teachers, professional development is hardly synonymous with learning content knowledge. Far too often, "professional development" is filled with games, fun new manipulatives, the latest pedagogical strategies, and classroom projects that supposedly make mathematics easy. The more serious kind of professional development, which some small percentage of teachers are lucky enough to participate in, addresses topics such as children's mathematical thinking, appropriate use of technology, teacher-student communication, and refined teaching practices. While these are important issues for teaching, they are not sufficient for transitioning from TSM to the CCSMS. Right now, professional development that replaces TSM with correct, coherent, precise, and logical

K–12 mathematics is urgently needed.

A natural reaction to the last point would be disbelief: don't colleges and universities teach future teachers the mathematics they need for teaching? Some may, but the vast majority do not (if they did, teachers would be continuously complaining about the errors in their students' textbooks, and our international ranking on mathematics assessments would be much higher).

In courses for future high school math teachers, colleges and universities usually teach university-level mathematics. The idea is that the "Intellectual Trickle-Down Theory" should work: learn advanced mathematics and you would automatically be knowledgeable about school mathematics. But it doesn't work, not in theory and not in practice. What colleges and universities should do is erase the damage done by TSM and revamp future high school teachers' knowledge of the algebra, geometry, trigonometry, etc., that they will be teaching.

In courses for future elementary teachers, who have to master a whole range of subjects, colleges and universities often teach pedagogy-focused "math methods" that merely embellish TSM.* These courses are usually taught by mathematics education professors, not mathematicians (who avoid teaching such courses because they wrongly see elementary mathematics as trivial); so it may well be that in most of these math methods courses no one—not even the professor—is aware of the flaws in TSM.

*Future teachers certainly do need to learn effective pedagogy, but they also must learn the content they will teach. This article is about building relevant and sound mathematics content knowledge into teacher preparation; it is not about taking pedagogical studies away from teacher preparation.

Perhaps we can better expose the absurdity of the way we prepare mathematics teachers if we consider the analogous situation of producing good high school French teachers: should we require them to learn Latin in college but not French? After all, Latin is the mother language of French and is linguistically more complex than French. Surely mastering a more complex language would enhance teachers' understanding of the French they already know from their school days. Is teaching future French teachers Latin any different from teaching future geometry teachers university-level mathematics? I don't think it is. In the same way, if we want to produce good elementary French teachers, wouldn't we ensure that they are fluent and literate in French before they begin courses on methods for teaching French? We would—and we should expect no less of our higher education institutions' approach to preparing elementary math teachers.

Right now, professional development that replaces Textbook School Math with correct, coherent, precise, and logical K–12 mathematics is urgently needed.

The failure of institutions of higher learning to take seriously their obligation to properly prepare mathematics teachers is a main reason why TSM has become entrenched in K–12.† The fail-

†This is not the only reason. The long-standing separation between educators and mathematicians is the other one.

The Fundamental Principles of Mathematics

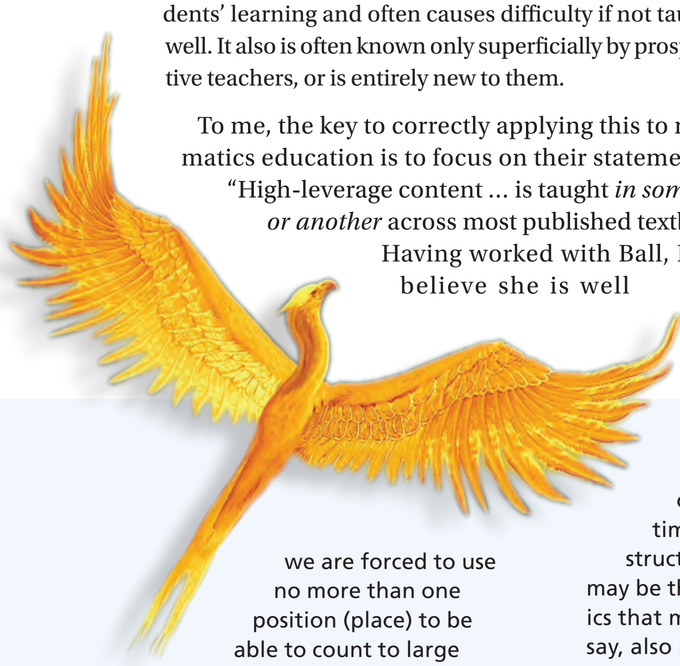
I believe there are five interrelated, fundamental principles of mathematics. They are routinely violated in school textbooks and in the math education literature, so teachers have to be aware of them to teach well.

1. *Every concept is precisely defined, and definitions furnish the basis for logical deductions*. At the moment, the neglect of definitions in school mathematics has reached the point at which many teachers no longer know the difference between a definition and a theorem. The general perception among the hundreds of teachers I have worked with is that a definition is "one more thing to memorize." Many bread-and-butter concepts of K–12 mathematics are not correctly defined or, if defined, are not put to use as integral parts of reasoning. These include number, rational number (in middle school), decimal (as a fraction in upper elementary school), ordering of fractions, product of fractions, division of fractions, length-area-volume (for different grade levels), slope of a line, half-plane of a line, equation, graph of an equation, inequality between functions, rational exponents of a positive number, polygon, congruence, similarity, parabola, inverse function, and polynomial.
2. *Mathematical statements are precise. At any moment, it is clear what is known and what is not known*. There are too many places in school mathematics in which textbooks and other education materials fudge the boundary between what is true and what is not. Often a heuristic argument is conflated with correct logical reasoning. For example, the identity $\sqrt{a}\sqrt{b} = \sqrt{ab}$ for positive numbers a and b is often explained by assigning a few specific values to a and b and then checking for these values with a calculator. Such an approach is a poor substitute for mathematics because it leaves open the possibility that there are other values for a and b for which the identity is not true.
3. *Every assertion can be backed by logical reasoning*. Reasoning is the lifeblood of mathematics and the platform that launches problem solving. For example, the rules of place value are logical consequences of the way we choose to count. By choosing to use 10 symbols (i.e., 0 to 9),

This sidebar is adapted with permission from "The Mis-Education of Mathematics Teachers" by Hung-Hsi Wu, which was published in the March 2011 issue of the Notices of the American Mathematical Society *(www.ams.org).*

ure occurs on two fronts: content knowledge and pedagogy. My main concern is with content knowledge, as I believe that mastering the mathematics is the hardest part of becoming a good math teacher, but I appreciate that others are focused on pedagogy. For example, an article in the last issue of *American Educator* by Deborah Loewenberg Ball and Francesca M. Forzani addresses the inadequacy of teachers' pedagogical preparation across all subjects.[4] I encourage readers who are interested in improving mathematics pedagogy to read their article. However, because Ball and Forzani are talking about teacher preparation in general, not just in mathematics, I would like to call attention to the following passage,[5] which may be misinterpreted in the context of *mathematics* teachers:

> "High-leverage content" comprises those texts, topics, ideas, and skills in each school subject area that are essential for a beginning teacher to know well. High-leverage content is foundational to the ideas and skills of the K–12 curricula in this country, is taught in some form or another across most published textbooks and curricula, and appears frequently.
>
> In addition, high-leverage content is fundamental to students' learning and often causes difficulty if not taught well. It also is often known only superficially by prospective teachers, or is entirely new to them.



To me, the key to correctly applying this to mathematics education is to focus on their statement that "High-leverage content ... is taught *in some form or another* across most published textbooks." Having worked with Ball, I believe she is well aware of the flaws in students' math textbooks. TSM does touch on all the important mathematics *in some form or another*, but almost never in a correct form. So while I would agree that high-leverage *topics* can be found in today's most widely used math textbooks, I would not agree that high-leverage *content* can be found in them.

That, of course, brings me back to my main concern. Because of the teacher preparation programs' failure to teach content knowledge relevant to K–12 classrooms, the vast majority of preservice teachers do not acquire a correct understanding of K–12 mathematics while in college. Because the flawed TSM they learned as K–12 students is not exposed, much less corrected, they unwittingly inflict TSM on their own students when they become teachers. So it comes to pass that TSM is recycled in K–12 from generation to generation. Today, this vicious cycle is so well ingrained that many current and future mathematics educators also are victimized by TSM, and their vision of K–12 mathematics is impaired. They have been led to equate TSM with "mathematics," so their educational commentaries on the school mathematics curriculum, by their implicit or explicit reference to TSM, become an unwitting affirmation of TSM. And so TSM lives on.

As a mathematician surveying this catastrophic education mess, I have to admit that, when all is said and done, the mathematics community has to take the bulk of the blame. We think school mathematics is too trivial,[6] and we think the politics of education is a bottomless pit not worthy of our attention. So we take the easy way out by ignoring all the goings-on in the schools and simply declare that if we teach high school teachers good mathematics, the rest is up to them. In other words, we hide behind the Intellectual Trickle-

we are forced to use no more than one position (place) to be able to count to large numbers.[‡] Given the too frequent absence of reasoning in school mathematics, how can we ask students to solve problems if teachers have not been prepared to engage students in logical reasoning on a consistent basis?

4. *Mathematics is coherent; it is a tapestry in which all the concepts and skills are logically interwoven to form a single piece*. The professional development of math teachers usually emphasizes either procedures (in days of yore) or intuition (in modern times), but not the coherent structure of mathematics. This may be the one aspect of mathematics that most teachers (and, dare I say, also math education professors) find most elusive. For instance, the lack of awareness of the coherence of the number systems in K–12 (whole numbers, integers, fractions, rational numbers, real numbers, and complex numbers) may account for teaching fractions as "different from" whole numbers such that the learning of fractions becomes almost divorced from the learning of whole numbers. Likewise, the resistance that some math educators (and therefore teachers) have to explicitly teaching children the standard algorithms may arise from not knowing the coherent structure that underlies these algorithms: the essence of all four standard algorithms is the reduction of any whole number computation to the computation of single-digit numbers.
5. *Mathematics is goal oriented, and every concept or skill has a purpose*. Teachers who recognize the purposefulness of mathematics gain an extra tool to make their lessons more compelling. For example, when students see the technique of completing the square merely as a trick to get the quadratic formula, rather than as the central idea underlying the study of quadratic functions, their understanding of the technique is superficial. Mathematics is a collection of interconnecting chains in which each concept or skill appears as a link in a chain, so that each concept or skill serves the purpose of supporting another one down the line. Students should get to see for themselves that the mathematics curriculum moves forward with a purpose.

–H.W.

[‡]For a thorough explanation of place value, please see "What's Sophisticated about Elementary Mathematics?," which I wrote for the Fall 2009 issue of *American Educator*, available at www.aft.org/pdfs/americaneducator/fall2009/wu.pdf.

Down Theory, even though we are daily confronted with evidence that it is not working.

Of course, some mathematicians have tried to make a contribution to school mathematics. But most of them have not devoted enough time to investigating the problem. They tend to be unaware of the sorry state of TSM and end up writing books that encourage teachers to *build on their knowledge of TSM* to solve problems or learn new mathematics.* This is akin to helping a starving person by buying him new clothes to make him look better without trying to address the malnutrition problem. With the opportunity provided by the CCSMS hovering over us, it is time that we mathematicians make amends.

*Unfortunately, this statement appears to hold true for almost all education writings in which mathematicians are involved.

In March of 2008, I was passing through London's Heathrow Airport and happened to catch sight of an ad by IBM:

> *Stop* selling what you have.
> *Start* selling what they need.

If we let "they" be our math teachers and math education professors, then this would be a pointed directive on what mathematicians need to do for school mathematics education:

> Get to know what *they* need, and teach it.

The advent of the CCSMS sends out the signal, for the first time from within the education community, that TSM has no place in the school curriculum. TSM is incompatible with the CCSMS, and now colleges and universities are duty-bound to provide future mathematics teachers with a replacement of TSM. Would that those

A University-Level Look at Adding Fractions and Multiplying Negative Numbers

In the main article, I argue that university-level mathematics courses tend to provide content that is mathematically sound but not relevant to the K–12 classroom. It may not be apparent that devising content knowledge that is both relevant and sound is a severe challenge, so let us consider Examples 1 and 2 (from the beginning of the main article) again to see how a typical university-level mathematics course would handle both problems.

What does the abstract mathematics of fractions have to say about adding $\frac{1}{8}$ to $\frac{5}{6}$? First of all, a fraction $\frac{m}{n}$ (for whole numbers m and n, $n \neq 0$) is just a symbol consisting of an ordered pair of whole numbers with m preceding n. It is just a symbol, with no mention of "parts of a whole" or "division." Two such ordered pairs $\frac{m}{n}$ and $\frac{k}{l}$ are considered to be equal if $ml = nk$. (In other words, the cross-multiplication algorithm is "declared" to be true.) In this context, how to add two such symbols becomes a matter of definition: we have to fashion a definition that will be consistent not only with the above meaning of equality but also with associative and commutative laws of addition. It was found that the definition of

$$\frac{m}{n} + \frac{k}{l} = \frac{ml + nk}{nl}$$

is satisfactory. So addition now becomes a concept created in a context of formal abstract mathematics. Then, of course,

$$\frac{1}{8} + \frac{5}{6} = \frac{(6 \times 1) + (8 \times 5)}{48} = \frac{46}{48}$$

as before. As to the problem of why (-2)(-3) = 2 × 3, the mathematical approach is to ignore integers but to prove once and for all that $(-x)(-y) = xy$ for all numbers x and y. Here is the proof:

> We first prove that $(-x)z = -(xz)$ for any numbers x and z. Observe that if a number A satisfies $w + A = 0$, then $A = -w$. Now if $A = (-x)z$, the distributive law implies $xz + A = xz + [(-x)z] = (x + (-x))z = 0 \cdot z = 0$. So indeed $(-x)z = -(xz)$. If we let $z = -y$ for a given y, this implies $(-x)(-y) = -(x(-y))$.
>
> Now let $B = (-x)(-y)$. To prove $B = xy$, it suffices to prove $xy - B = 0$. This is so because $xy - B = xy - [-(x(-y))] = xy + x(-y) = x[y + (-y)] = x \cdot 0 = 0$, as desired.

Neither of the above solutions would be usable in school classrooms. Teaching this kind of mathematics to teachers may serve *some* purpose, but not the purpose of helping them to teach their lessons. Take the mathematical proof of $(-x)(-y) = xy$ for *all* numbers x and y, for example. It is not suitable for school use, either by teachers or students, because students in middle school are still fully immersed in arithmetic; their natural habit is to find out what a number is by direct computations. This proof of $(-x)(-y) = xy$ is all about abstract, indirect reasoning. At their stage of mathematical development, middle school students are not yet used to thinking in such abstract generality. Such a proof, therefore, simply fails to make contact with their mathematical sensibilities. For this reason, the approach described in the main article to first prove it for (-2)(-3), and then $(-m)(-n)$ for whole numbers m and n, is nothing more than an attempt to narrow the gap between students' background in arithmetic and the abstraction inherent in the reasoning. It changes the discourse about arbitrary fractions to whole numbers—a subject students are comfortable with—and it makes use of the familiar skill of *counting* as part of the reasoning, e.g., -3 = (-1) + (-1) + (-1). Thus the abstraction has been modified for students' consumption.

–H.W.

institutions were aware of their duties! Teachers of all levels now must learn to teach mathematics, not just with analogies and metaphors, and not just with incomprehensible pseudo-explanations and decrees, but with precision, reasoning, and coherence.

Mathematical Engineering

It will not be enough for institutions of higher learning to teach future teachers rigorous advanced mathematics, because the topics in school mathematics are not part of advanced mathematics. Nor will it suffice to pass off pedagogy-laden courses as mathematics courses, because the mathematical difficulties that lead to nonlearning cannot be cured with pedagogical techniques. That said, the pressing need now is to provide all future mathematics teachers with content knowledge that satisfies both of the following requirements:

Preparing to teach proper school mathematics is not about learning a craft but, rather, a discipline that is cognitively complex and hierarchical. Each topic, no matter how basic, is essential to some future topic.

A. It is relevant to teaching—i.e., does not stray far from the material they teach in school.
B. It is consistent with the following five fundamental principles of mathematics: precise definitions are the basis for logical deductions; precise statements clarify what is known and what is not known; every assertion can be backed by logical reasoning; all the concepts and skills are woven together like a tapestry; and each concept and skill has a purpose. (I briefly explain each of these in the box on pages 8–9.)

Currently, TSM satisfies requirement A, at least in the sense that it attempts to "cover" all of the mathematics needed in K–12 (however, it is also riddled with unnecessary topics—but that is another article). But TSM does not satisfy requirement B at all. University-level mathematics satisfies B, but not A. (Those who are not convinced should read the box on page 10.) What we are witnessing, therefore, is two extremes in the presentation of mathematics, each one satisfying one of the two conditions but not the other.

The middle ground—which must be both accessible to children and mathematically correct—is a modified or *customized* version of university-level mathematics. Examples 1 and 2 above provide illustrations of such customization.

This brings us to a clearer conception of what K–12 mathematics education is all about: *mathematical engineering*, in the sense that it is a customization of abstract, university-level mathematics for the consumption of school students. Let us put this in context. Engineering is the discipline of customizing abstract scientific principles into processes and products that safely realize a human objective or function. So, chemical engineering begins with chemistry and results in Plexiglas tanks in aquariums, the gas you pump into your car, shampoo, Lysol, etc. Electrical engineering transforms the abstract theory of electromagnetism into computers, iPods, lights in your hall, hybrid motors, etc. And in the same vein, mathematical engineering takes abstract, university-level mathematics and customizes it into *school mathematics* (distinct from TSM) that can be correctly taught, and learned, in K–12 classrooms.

My hope is that the CCSMS will usher in mathematical engineering, drive out TSM, and replace TSM with school mathematics proper.[7] But if our mathematical engineering work is limited to standards and large-scale assessments (which, sadly, seems to be where we are currently headed), then nothing will be accomplished. Proper school mathematics textbooks for teachers and students, model lesson plans, diagnostic assessments, and professional development are absolutely necessary. These things are often discussed as instructional "supports," implying that only weak teachers would need them. That is absurd. Is it only the weak chemists who need proper lab equipment or only the weak basketball players who work with coaches?

To do all the necessary mathematical engineering work well, mathematicians, mathematics education professors, and mathematics teachers must work together. These groups' history of working independently has given us inadequate TSM for students, pedagogy-focused math-light courses for future elementary teachers, and irrelevant university-level math courses for future high school math teachers. If these groups came together, they would finally have the knowledge of mathematics, children, pedagogy, and classroom realities necessary to replace TSM with proper school mathematics, and to create rigorous and relevant math courses for future (and current) teachers. (The federal agencies that have followed the development of the CCSMS should take note of this need and provide financial incentives for the reconciliation.)

There are two major impediments to this work: a shortage of willing mathematicians, and a shortage of teachers and mathematics education professors who realize that TSM is inadequate. As a mathematician who has worked with K–12 teachers for more than a decade, I believe the latter shortage will be much easier to address than the former. Most of the hundreds of teachers I have worked with are eager to improve, and they are relieved to discover that their own difficulties with mathematics are a result of the TSM they have been taught. In addition, once we have made progress in our mathematical engineering, teacher preparation can be completely overhauled, and the vicious cycle that perpetuates TSM will be broken. But first, we must address the shortage of willing mathematicians. I have a radical proposal: professional mathematics organizations, especially the American Mathematical Society, should sponsor training for a new corps of competent mathematicians to get to know the school mathematics curriculum and then dedicate themselves to mathematical engineering. Like chemical and electrical engineering, mathematical engineering ought to become an established interdisciplinary discipline.

Assuming the work of mathematical engineering gets going, we will still face a few additional obstacles. First, district leadership will have to comprehend that teaching this new proper school mathematics to in-service teachers requires a long-term commitment. Learning mathematics, and unlearning TSM, will take effort and time. Two or three half-day sessions each semester will not be sufficient. In mathematics, the most difficult part of a teacher's professional development is the acquisition of solid content knowledge. Preparing to teach proper school mathematics is not about learning a craft; rather, it is about learning a discipline that is cognitively complex and very hierarchical. Each topic, no matter how basic, is essential to some future topic. For example, understanding place value is essential to understanding multidigit addition, and understanding multiplication of fractions is essential to understanding algebra, etc.

Second, although I would like nothing more than for all of the nation's elementary-grades teachers to be immersed in the intensive school mathematics education that they should have received in college, two things work against that: the fact that there are more than 1.5 million elementary teachers, and the fact that they are required to teach all subjects. Expecting any one person to expertly teach reading, mathematics, and all other subjects is just wishful thinking masquerading as national policy. A more sensible approach would be to have *mathematics teachers* in elementary school.[8] (To read more about this idea, please see "What's Sophisticated about Elementary Mathematics? Plenty—That's Why Elementary Schools Need Math Teachers," which I wrote for the Fall 2009 issue of *American Educator*, available at www.aft.org/pdfs/americaneducator/fall2009/wu.pdf.)

A third potential obstacle is the assessment that comes with the CCSMS. State officials should be vigilant in safeguarding their students from being overtested. They must remember that while *some* standardized assessment is necessary and healthy, several assessments a year would be counterproductive to learning. Another concern is about the mathematical quality of test items.

Understanding Numbers in Elementary School Mathematics

A New Textbook for Teachers

Knowing that most K–12 teachers do not receive adequate professional development (either pre-service or in-service) on the mathematics content that they must teach, Hung-Hsi Wu has spent more than a decade conducting intensive, three-week summer institutes for teachers. Now, he has taken what he has learned from his students (i.e., hundreds of teachers) and written a mathematics textbook for teachers in grades K–6. It's not an instructional guide or a suggested curriculum or a set of model lesson plans; it's a mathematics textbook. Although it requires, as Wu writes, "serious effort," it delivers the mathematical knowledge that elementary-grades teachers need—starting with place value (literally, "How to Count") and ending with decimal expansions of fractions. To provide an overview of the textbook, and of the volumes to come for middle and high school teachers, the following is an excerpt from the preface.

–EDITORS

How does this textbook differ from textbooks written for students in K–6? The most obvious difference is that, because adults have a longer attention span and a higher level of sophistication, the exposition of this book is more concise; it also offers coherent logical arguments instead of sound bites. Because the present consensus is that math teachers should know the mathematics beyond the level they are assigned to teach,* this book also discusses topics that may be more appropriate for grades 7 and 8. Because teachers also have to answer questions from students, some of which can be quite profound, their knowledge of what they teach must go beyond the minimal level. Ideally, they should know mathematics in the sense that mathematicians use the word "know": *knowing* a concept means knowing its precise definition, its intuitive content, why it is needed, and in what contexts it plays a role, and *knowing* a skill means knowing precisely what it does, when it is appropriate to apply it, how to prove that it is correct, the motivation for its creation, and, of course, the ability to use it correctly in diverse situations. For this reason, this book tries to provide such needed information so that teachers can carry out their duties in the classroom.

The most noticeable difference between this book and student texts is, however, its comprehensive and systematic *mathematical* development of the numbers that are the bread and butter of the K–12 curriculum: whole numbers, fractions, and rational numbers. Such a

*See Recommendation 19 on page xxi in *Foundations for Success: The Final Report of the National Mathematics Advisory Panel*, www2.ed.gov/about/bdscomm/list/mathpanel/report/final-report.pdf.

At the moment, students' need of a mathematically valid assessment is undercut by the presence of flawed and mathematically marginal items in standardized tests, including those from NAEP.[9] To minimize such errors in the future, we need assurance from both of the assessment consortia that they are committed to getting substantive and continuing input from competent mathematicians.

Our nation has been known to overcome greater obstacles than these, provided the cause is worthy. Because failure in math education has far-reaching consequences,[10] the worthiness of successfully implementing the CCSMS is clear. Furthermore, the CCSMS are likely our last hope of breaking the vicious cycle of TSM for a long time to come. Can we all contribute our share to make sure that the CCSMS will stay the course?

Our children are waiting for an affirmative answer. ☐

Endnotes

1. For further discussion, see H. Wu, "The Impact of Common Core Standards on the Mathematics Education of Teachers," April 29, 2011, http://math.berkeley.edu/~wu/CommonCoreIV.pdf.

2. H. Wu, "What Is Different about the Common Core Mathematics Standards?" June 20, 2011, http://math.berkeley.edu/~wu/CommonCoreVI.pdf.

3. National Mathematics Advisory Panel, "Chapter 3: Report of the Task Group on Conceptual Knowledge and Skills" (Washington, DC: U.S. Department of Education, 2008), page 3-63.

4. Deborah Loewenberg Ball and Francesca M. Forzani, "Building a Common Core for Learning to Teach," *American Educator* 35, no. 2 (Summer 2011): 17–21 and 38–39.

5. Ball and Forzani, "Building a Common Core," 38.

6. But it is not. See the preface of Wu's book, *Understanding Numbers in Elementary School Mathematics* (Providence, RI: American Mathematical Society, 2011).

7. See H. Wu, "How Mathematicians Can Contribute to K–12 Mathematics Education," February 26, 2006, http://math.berkeley.edu/~wu/ICMtalk.pdf, for further discussion.

8. See, for example, pages 5-51 to 5-58 of National Mathematics Advisory Panel, "Chapter 5: Report of the Task Group on Teachers and Teacher Education" (Washington, DC: U.S. Department of Education, 2008).

9. See pages 8-3 to 8-4 of National Mathematics Advisory Panel, "Chapter 8: Report of the Task Group on Assessment" (Washington, DC: U.S. Department of Education, 2008).

10. See, for example, *Rising Above the Gathering Storm, Revisited: Rapidly Approaching Category 5* (Washington, DC: National Academies Press, 2010), www.nap.edu/catalog.php?record_id=12999.

development acquires significance in light of the recent emphasis on *mathematical coherence* in educational discussions. Coherence in mathematics is not something ineffable like Mona Lisa's smile. It is a quality integral to mathematics with concrete manifestations affecting every facet of mathematics. If we want a coherent curriculum and a coherent progression of mathematics learning, we must have at least one default model of a logical, coherent presentation of school mathematics that respects students' learning trajectory. It is unfortunately the case that, for a long time, such a presentation has not been readily available. The mathematics community has been derelict in meeting this particular social obligation.

If we want a coherent curriculum, we must have at least one default model of a logical, coherent presentation of school mathematics that respects students' learning trajectory.

This book does not call attention to coherence per se, but tries instead to demonstrate coherence by example. Its systematic mathematical development makes it possible to point out the careful logical sequencing of the concepts and the multiple interconnections, large and small, among the concepts and skills.† Thus, it points out the fact that the usual algorithm for converting a fraction to a decimal by long division, if done correctly, is in fact a consequence of the product formula for fractions, $\frac{m}{n} \times \frac{k}{l} = \frac{mk}{nl}$. It also points out the overwhelming importance of the theorem on equivalent fractions (i.e., $\frac{m}{n} = \frac{cm}{cn}$) for the understanding of every aspect of fractions. On a larger scale, one sees in this systematic development the *continuity* in the evolution of the concepts of addition, subtraction, multiplication, and division from whole numbers to fractions, to rational numbers, and finally—in the context of school mathematics—to real numbers. Although each arithmetic operation may look superficially different in different contexts, this book explains why it is fundamentally the same concept throughout. Thus, with a systematic development in place, one can step back to take a global view of the entire subject of numbers and gain some perspective on how the various pieces fit together to form a whole fabric. In short, such a development is what gives substance to any discussion of coherence.

This book is one mathematician's attempt at a systematic presentation of the mathematics of K–6. It is the product of more than 10 years of experimentation in my effort to teach mathematics to elementary and middle school teachers. The starting point was the workshop on fractions that I conducted in March of 1998. Subsequent volumes written for middle school and high school teachers will round out the curriculum of the remaining grades. My fervent hope is that others will carry this effort further so that we can achieve an overhaul of the mathematical education of teachers as we know it today. Our teachers deserve better, and our children deserve no less.

–H.W.

†One should not infer from this statement that the systematic development presented in this book is the only one possible. This book follows the most common school model of going from whole numbers to fractions and then to rational numbers, but it would be equally valid, for example, to go from whole numbers to integers and then to rational numbers.

Understanding Numbers in Elementary School Mathematics, by Hung-Hsi Wu, is published by the American Mathematical Society (AMS). While the book was originally listed for $79, the AMS has it on sale for $47.40 through the end of 2011. To order, go to **www.ams.org/bookstore-getitem/item=MBK-79**.